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On skew-Hadamard matrices

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Dedicated to Professor Jennifer Seberry on the occasion of her 60th birthday

Abstract

Skew-Hadamard matrices are of special interest due to their use, among others, in constructing orthogonal designs. In this paper, we give a survey on the existence and equivalence of skew-Hadamard matrices. In addition, we present some new skew-Hadamard matrices of order 52 and improve the known lower bound on the number of the skew-Hadamard matrices of this order.

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1. Introduction

A $(1, -1)$ matrix H of order n is called *Hadamard matrix* if $HH^T = H^TH = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n . A $(1, -1)$ matrix A of order n is said to be of skew type if $A - I_n$ is skew-symmetric. A Hadamard matrix is normalized if all entries in its first row and column are equal to 1. A skew-Hadamard matrix H of order n can always be written in a skew-normal form as

$$H = \begin{pmatrix} 1 & e^T \\ -e & C + I \end{pmatrix},$$

where e^T is the $1 \times (n - 1)$ vector of ones, i.e. $e^T = (1, \dots, 1)$, and C is a skew-symmetric $(0, 1, -1)$ matrix.

Two Hadamard matrices are said to be *equivalent* if one can be transformed into the other by a series of row or column permutations and negations. It is well known that if n is the order of a Hadamard matrix then n is necessarily 1, 2 or a multiple of 4. However, it still remains open if a Hadamard matrix of order n exists for every $n \equiv 0 \pmod{4}$. Since a Hadamard matrix of order 428 is found recently (see [22]), the smallest order for which a Hadamard matrix is not yet known is $n = 668$.

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Skew-Hadamard matrices are of great interest (see [32]) because of their elegant structure, their beautiful properties and their application in constructing orthogonal designs, D-optimal weighing designs for $n \equiv 3 \pmod{4}$ (see for example [24]), and edge designs (see for example [12,18,19]).

In this survey article, we discuss the existence and the equivalence of skew-Hadamard matrices. We present some known construction methods of skew-Hadamard matrices, the known results and also some new inequivalent skew-Hadamard matrices of order 52. As a consequence, we improve the known lower bound of the number of inequivalent skew-Hadamard matrices of this order from one [2] to 561.

2. Construction methods

Many constructions of Hadamard matrices are known but not all of them give skew-Hadamard matrices. In this section we recall some well known methods for constructing skew-Hadamard matrices.

Theorem 1 (Wallis et al. [38, Theorem 4.1]). *Let t be a non-negative integer and $k_i \equiv 0 \pmod{4}$ be a prime power plus one, for each i . Then there is a skew-Hadamard matrix of order $2^t \prod k_i$.*

Theorem 2 (Wallis et al. [38, Theorem 4.2]). *Let u be any odd integer and h the order of a skew-Hadamard matrix. Then there is a skew-Hadamard matrix of order $(h-1)^u + 1$.*

In order to explain the following constructions we need the definition of supplementary difference sets (sds) (see [8,29]).

Definition 1. Let S_1, S_2, \dots, S_n be subsets of V , an additive abelian group of order v . These subsets are called $n - \{v; k_1, k_2, \dots, k_n; \lambda\}$ sds modulo n if $|S_t| = k_t$ for $t = 1, 2, \dots, n$ and for each $m \in \{1, 2, \dots, v-1\}$ we have $\lambda_1(m) + \dots + \lambda_n(m) = \lambda$, where $\lambda_t(m)$ is the number of solutions (i, j) of the congruence $i - j \equiv m \pmod{v}$ with $i, j \in S_t$.

2.1. Constructions using Szekeres difference sets

More details on these constructions can be found in [36,38].

Definition 2. Let G be an additive abelian group of order $2m+1$. Then two subsets $A \subset G, B \subset G$, each of size m , will be called *Szekeres difference sets* if

- (i) $a \in A \Rightarrow -a \notin A$, and
- (ii) for each $d \in G, d \neq 0$, the total number of solutions $(a_1, a_2) \in A \times A, (b_1, b_2) \in B \times B$ of the equations $d = a_1 - a_2, d = b_1 - b_2, d = b_1 - b_2$, is $m-1$.

Alternatively, $2 - \{2m+1; m, m; m-1\}$ supplementary difference sets A and B are called *Szekeres difference sets* if $a \in A \Rightarrow -a \notin A$.

Theorem 3. *If A and B are two Szekeres difference sets of size m in an additive abelian group of order $2m+1$, then there is a skew-Hadamard matrix of order $4(m+1)$.*

Theorem 4. *Let $q = 2m+1 = p^r \equiv 5 \pmod{8}$ be a prime power and let G be the elementary abelian group of order p^k . Then there exist Szekeres difference sets of size m in G and a skew-Hadamard matrix of order $2(q+1)$.*

The following theorem was proved independently by Szekeres [36] and Whiteman [39].

Theorem 5. *If $q = p^t = 8m+1$ is a prime power such that $p \equiv 5 \pmod{8}$ and $t \equiv 2 \pmod{4}$, then there exist Szekeres difference sets of size $4m$ and a skew-Hadamard matrix of order $2(q+1)$.*

2.2. The Williamson construction

Theorem 6 (Wallis et al. [38]). Let A, B, C and D be square matrices of order n . Further, let A be skew-type and circulant and B, C, D be back-circulant matrices whose first rows satisfy the following equations:

$$\left. \begin{aligned} a_{1,j} &= -a_{1,n+2-j} \\ b_{1,j} &= b_{1,n+2-j} \\ c_{1,j} &= c_{1,n+2-j} \\ d_{1,j} &= d_{1,n+2-j} \\ a_{11} &= b_{11} = c_{11} = d_{11} = +1 \end{aligned} \right\} \quad 2 \leq j \leq n, \quad (1)$$

where $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ and every element is $+1$ or -1 . If

$$AA^T + BB^T + CC^T + DD^T = 4nI_n,$$

then

$$H = \begin{pmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{pmatrix} \quad (2)$$

is a skew-Hadamard matrix of order $4n$.

2.3. The Goethals–Seidel construction

Let G be an additive abelian group of order n with elements g_1, g_2, \dots, g_n and X a subset of G . Define the type $1(1, -1)$ incidence matrix $M = (m_{ij})$ of order n of X to be

$$m_{ij} = \begin{cases} +1 & \text{if } g_j - g_i \in X, \\ -1 & \text{otherwise,} \end{cases}$$

and the type 2 $(1, -1)$ incidence matrix $N = (n_{ij})$ of order n of X to be

$$n_{ij} = \begin{cases} +1 & \text{if } g_j + g_i \in X, \\ -1 & \text{otherwise.} \end{cases}$$

In particular, if G is cyclic the matrices M and N are circulant and back circulant, respectively. In this case $m_{ij} = m_{1,j-i+1}$ and $n_{ij} = n_{1,i+j-1}$, respectively, (indices should be reduced modulo n).

Theorem 7 (Goethals and Seidel [21] or Wallis et al. [38]). Suppose there exist four circulant $(1, -1)$ (or type I) matrices A, B, C, D of order n satisfying

$$AA^T + BB^T + CC^T + DD^T = 4nI_n. \quad (3)$$

Let R be the back diagonal matrix. Then the Goethals–Seidel array

$$GS = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & -D^T R & C^T R \\ -CR & D^T R & A & -B^T R \\ -DR & -C^T R & B^T R & A \end{pmatrix} \quad (4)$$

is a Hadamard matrix of order $4n$. Furthermore, if A is skew-type then GS is a skew-Hadamard matrix.

Some known classes of four matrices that can be used in the Goethals–Seidel construction are described in the following:

Four $(1, -1)$ matrices A, B, C, D of order n (odd) with the properties:

- (i) $MN^T = NM^T$ for $M, N \in \{A, B, C, D\}$;
- (ii) $(A - I)^T = -(A - I), B^T = B, C^T = C, D^T = D$;
- (iii) $AA^T + BB^T + CC^T + DD^T = 4nI_n$

will be called *good matrices*. These are found and used, among other places, in [17,27,31,37]. If instead of (i) and (ii) we have

- (i)' $MN = NM$ for $M, N \in \{A, B, C, D\}$,
- (ii)' $(A - I)^T = -(A - I), (B - I)^T = -(B - I), C^T = C, D^T = D$,

then these matrices are called *G-matrices*. Such matrices were first introduced and applied to construct skew-Hadamard matrices by Seberry Wallis in [30], and studied further in [10,16,23,41]. Spence [35] has constructed an infinite family of G-matrices. Their orders are $n = (q + 1)/2$ where q is a prime power $\equiv 5 \pmod{8}$. The first 10 such numbers are $n = 3, 7, 15, 19, 27, 31, 51, 55, 63, 75$. If instead of (i) and (ii) we have (i)' and

- (ii)'' $(A - I)^T = -(A - I), (B - I)^T = -(B - I), (C - I)^T = -(C - I), D^T = D$

then these matrices are called *best matrices*. Such matrices have been recently introduced and applied to construct skew-Hadamard matrices in [16]. We also need the definition of the quadratic character χ on the elements of $GF(p^n)$, p prime. That is,

$$\chi(x) = \begin{cases} 0, & x = 0, \\ 1, & x = y^2, \text{ for some } y \in GF(p^n), \\ -1 & \text{otherwise.} \end{cases}$$

Theorem 8 (Whiteman, [40] or [38, Theorem 4.15]). Let q be a prime power $\equiv 3 \pmod{8}$ and put $n = (q + 1)/4$. Let γ be a primitive element of $GF(q^2)$. Put $\gamma^k = ax + b$ ($a, b \in GF(q)$) and define $a_k = \chi(a), b_k = \chi(b)$. Let A, B, C, D be square circulant matrices of order n whose initial rows are given by $a_0, a_8, a_{16}, \dots, a_{8(n-1)}; b_0, b_8, b_{16}, \dots, b_{8(n-1)}; a_1, a_9, a_{17}, \dots, a_{8n-7}; b_1, b_9, b_{17}, \dots, b_{8n-7}$; respectively. Then the matrix H defined by (4) is a skew-Hadamard matrix of order $4n$.

2.4. The Wallis–Whiteman construction

The theorems given in this section can be found in [33,38]. Theorem 9 is equivalent to Theorem 7 but they were discovered independently and under different circumstances.

Theorem 9. Suppose X, Y and W are type 1 incidence matrices and Z is a type 2 incidence matrix of $4 - \{v; k_1, k_2, k_3, k_4; \sum_{i=1}^4 k_i - v\}$ supplementary difference sets. Then if

$$A = 2X - J, \quad B = 2Y - J, \quad C = 2Z - J, \quad D = 2W - J,$$

$$H = \begin{pmatrix} A & B & C & D \\ -B^T & A^T & -D & C \\ -C & D^T & A & -B^T \\ -D^T & -C & B & A^T \end{pmatrix}$$

is a Hadamard matrix of order $4v$. Furthermore, if A is skew-type then H is a skew-Hadamard matrix of order $4v$.

Theorem 10. Suppose X, Y and W are type 1 incidence matrices and Z is a type 2 incidence matrix of $4 - \{2m + 1; m, m, m, m; 2(m - 1)\}$ supplementary difference sets. Then, if

$$A = 2X - J, \quad B = 2Y - J, \quad C = 2Z - J, \quad D = 2W - J$$

we have that

$$H = \begin{pmatrix} -1 & -1 & -1 & -1 & e^T & e^T & e^T & e^T \\ 1 & -1 & 1 & -1 & -e^T & e^T & -e^T & e^T \\ 1 & -1 & -1 & 1 & -e^T & e^T & e^T & -e^T \\ 1 & 1 & -1 & -1 & -e^T & -e^T & e^T & e^T \\ e & e & e & e & A & B & C & D \\ -e & e & -e & e & -B^T & A^T & -D & C \\ -e & e & e & -e & -C & D^T & A & -B^T \\ -e & -e & e & e & -D^T & -C & B & A^T \end{pmatrix}$$

is a Hadamard matrix of order $8(m+1)$. Furthermore, if A is skew-type then H is a skew-Hadamard matrix of order $8(m+1)$.

Theorem 11. Let $q = 2m + 1 = 8f + 1$ (f odd) be a prime power. Then there exist $4 - \{2m + 1; m, m, m, m; 2(m-1)\}$ supplementary difference sets X_1, X_2, X_3, X_4 for which $y \in X_i \Rightarrow -y \notin X_i, i = 1, 2, 3, 4$.

Corollary 1. Let $q = 8f + 1$ (f odd) be a prime power. Then there exists a skew-Hadamard matrix of order $4(q+1)$.

2.5. Construction using amicable matrices

In order to provide two methods for multiplying the order of skew-Hadamard matrices we need the notion of amicable Hadamard matrices [20, p. 252]. This was first realized by Seberry Wallis in [26]. We recall that $M = I + S$ and N are amicable Hadamard matrices of order m if

- (i) $S^T = -S, SS^T = (m-1)I_m$;
- (ii) $N^T = N, NN^T = mI_m$;
- (iii) $MN^T = NM^T$.

Using amicable Hadamard matrices we can give the next multiplication theorem.

Theorem 12. Let m and m' be the orders of amicable Hadamard matrices. Then if there is a skew-Hadamard matrix of order $(m-1)m'/m$ there is a skew-Hadamard matrix of order $m'(m'-1)(m-1)$.

Theorem 13. Let h be the order of a skew-Hadamard matrix and m the order of amicable Hadamard matrices. Then there is a skew-Hadamard matrix of order mh .

More details on constructing skew-Hadamard matrices using amicable matrices can be found in [38].

2.6. Doubling construction

This construction method was first given in [27] where it was used for the construction of a skew-Hadamard matrix of order 184 using the skew-Hadamard matrix of order 92 which is also constructed in the same paper.

Theorem 14. Suppose that $H_n = S + I_n$ is a skew-Hadamard matrix of order n . Then

$$H_{2n} = \begin{pmatrix} S + I_n & S + I_n \\ S - I_n & -S + I_n \end{pmatrix}$$

is a skew-Hadamard matrix of order $2n$.

The smallest nontrivial order for which a Hadamard matrix exists is 2. A skew-Hadamard matrix of order 2 is

$$H_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = S_2 + I_2.$$

The *doubling construction* can be used to give an infinite family of skew-Hadamard matrices $H_{2^n} = S_{2^n} + I_{2^n}$ of order 2^n using

$$H_{2^n} = \begin{pmatrix} S_{2^{n-1}} + I_{2^{n-1}} & S_{2^{n-1}} + I_{2^{n-1}} \\ S_{2^{n-1}} - I_{2^{n-1}} & -S_{2^{n-1}} + I_{2^{n-1}} \end{pmatrix}, \tag{5}$$

for all $n \geq 1$.

2.7. Paley construction

One more infinite class of skew-Hadamard matrices can be found in [25]. This construction is called the *Paley construction* and can be used to give skew-Hadamard matrices of order $n = q + 1$, when $q \equiv 3 \pmod{4}$ is a prime power.

3. Known results

There are two very tough problems concerning skew-Hadamard matrices. The first being the existence and the construction of such matrices. Although this problem has been widely studied by many researchers [5–9,16–18,21,31,37,39], there are a lot of orders for which skew-Hadamard matrices have not been constructed yet. Recently, a skew-Hadamard matrix of order 236 has been constructed in [13], and of orders 188 and 388 in [11]. The first unsettled case corresponds to order $276 = 69 \cdot 4$. The current status on known results and open problems on the existence of skew-Hadamard matrices of order $2^t m$, m odd, $m < 500$, are given in Table 1, which is an update of Table 24.31 given in [4]. In Table 1, we write $m(t)$ if the skew-Hadamard matrix of order $2^t m$ exists. An $m(\cdot)$ means that a skew-Hadamard matrix of order $2^t m$ is not yet known for any t . The values $m < 500$, missing from Table 1, indicate that a skew-Hadamard matrix of order $4m$ exists. Seberry Wallis [28] conjectured that skew-Hadamard matrices exist for all dimensions divisible by 4.

The second problem is to find the exact number N_n of inequivalent skew-Hadamard matrices for a given order n . This problem has attracted a lot of attentions lately. In Table 2, we denote by n the order of the skew-Hadamard matrix and

Table 1
Existence of skew-Hadamard matrices

	69(3)	89(4)		101(10)	107(10)	109(9)	119(4)
145(5)	149(4)	153(3)	167(4)	177(12)	179(8)	191(·)	193(3)
201(3)	205(3)	209(4)	213(4)	223(3)	225(4)	229(3)	233(4)
235(3)	239(4)	245(4)	247(4)	249(4)	251(6)	253(4)	257(4)
259(5)	261(3)	265(4)	269(8)	275(4)	277(5)	283(11)	285(3)
287(4)	289(3)	295(5)	299(4)	301(3)	303(3)	305(4)	309(3)
311(26)	317(6)	319(3)	325(5)	329(6)	331(3)	335(7)	337(18)
341(4)	343(6)	345(4)	347(18)	349(3)	353(4)	359(4)	361(3)
369(4)	373(7)	377(6)	385(3)	389(15)	391(4)	397(5)	401(10)
403(5)	409(3)	413(4)	419(4)	423(4)	429(3)	433(3)	435(4)
441(3)	443(6)	445(3)	449(·)	451(3)	455(4)	457(9)	459(3)
461(17)	465(3)	469(3)	473(5)	475(4)	479(12)	481(3)	485(4)
487(5)	489(3)	491(46)	493(3)				

Table 2
Inequivalent skew-Hadamard matrices

n	2	4	8	12	16	20	24	28	32	36	40	44	48	52
N_n	1	1	1	1	2	1	16	54	≥ 6	≥ 18	≥ 22	≥ 59	≥ 1	≥ 561

Table 3
Skew-Hadamard matrices constructed from four circulant matrices of order n

n	h_n	Type-construction	n	h_n	Type-construction
3	1	Good matrices [37]	33	15	Good matrices [17]
	1	G-matrices [14]		31	G-matrices [14]
	1	Best matrices [16]	35	6	Good matrices [17]
5	1	Good matrices [37]	37	2	Good matrices [17]
	1	G-matrices [14]		≥ 4	G-matrices [14]
7	3	Good matrices [37]	39	5	Good matrices [17]
	1	G-matrices [14]	41	≥ 4	G-matrices [14]
	1	Best matrices [16]	43	≥ 1	Goethals–Seidel type [5]
9	1	Good matrices [37]	49	≥ 12	Goethals–Seidel type [6]
	2	G-matrices [14]	51	≥ 1	G-matrices [35]
11	3	Good matrices [37]	55	≥ 1	G-matrices [35]
13	6	Good matrices [37]	63	≥ 1	G-matrices [35]
	8	G-matrices [14]	65	≥ 13	Goethals–Seidel type [6]
	2	Best matrices [16]	67	≥ 1	Goethals–Seidel type [7]
15	11	Good matrices [37]	75	≥ 1	G-matrices [35]
	32	G-matrices [14]	81	≥ 2	Goethals–Seidel type [9]
17	2	Good matrices [37]	93	≥ 4	Goethals–Seidel type [6]
19	8	Good matrices [37]	103	≥ 1	Goethals–Seidel type [9]
	9	G-matrices [14]	113	≥ 1	Goethals–Seidel type [7]
21	10	Good matrices [37]	121	≥ 5	Goethals–Seidel type [6]
	23	G-matrices [14]	127	≥ 1	Goethals–Seidel type [7]
	21	Best matrices [16]		≥ 1	Good matrices [8]
23	6	Good matrices [37]	129	≥ 2	Goethals–Seidel type [6]
	16	G-matrices [14]	133	≥ 9	Goethals–Seidel type [6]
	9	Best matrices [16]	151	≥ 3	Goethals–Seidel type [9]
25	9	Good matrices [37]	157	≥ 1	Goethals–Seidel type [7]
	13	G-matrices [14]	163	≥ 1	Goethals–Seidel type [7]
27	12	Good matrices [37]	169	≥ 2	Goethals–Seidel type [9]
	20	G-matrices [14]	181	≥ 1	Goethals–Seidel type [7]
29	5	Good matrices [37]	217	≥ 4	Goethals–Seidel type [6]
31	3	Good matrices [37]	219	≥ 16	Goethals–Seidel type [6]
	8	G-matrices [14]	241	≥ 1	Goethals–Seidel type [7]
	8	Best matrices [16]	267	≥ 8	Goethals–Seidel type [6]
			463	≥ 1	Goethals–Seidel type [9]

by N_n the number of known inequivalent skew-Hadamard matrices of order n . For $n = 4, 8, 12, 20, 24, 28, 32, 44, 48$ the conditions of Paley's theorem are satisfied and thus we can construct a skew-Hadamard matrix of order n . Ito has determined that for general skew-Hadamard matrices, there is a unique matrix of each order less than 16, two of order 16, and 16 of order 24. H. Kimura has found 49 of order 28 and 6 of order 32, see [34, p. 492]. Later Baartmans, Lin and Wallis [1] improved the lower bound found by Kimura for order 28, showing that there are exactly 54 inequivalent skew-Hadamard matrices of order 28. 22 skew-Hadamard matrices of order 40 were constructed in a recent paper (see [15]). Results for orders 36, 44 have been recently given in [19,18], respectively. A skew-Hadamard matrix of order 52 was first found in [2]. New results for $n = 52$ are presented in Section 4.

In Table 3, we summarize the known results on good, G, best, and Goethals–Seidel type matrices of order n that can be used to construct skew-Hadamard matrices of order $4n$. By h_n , we denote the number of known inequivalent matrices of the corresponding type.

4. New skew-Hadamard matrices of order 52

In this section, we present the new skew-Hadamard matrices of order 52 that we have found. These new 561 matrices have been found using the following method. We constructed as many matrices as possible and then used some of the known criteria to check for the equivalence of these matrices. We decided to use the classical Goethals–Seidel array, which is one of the most famous construction methods. For this purpose and using a simple program written in Delphi ver. 5.0, we constructed many circulant matrices of order 13 satisfying (3). Then, using these 4-tuples of circulant matrices in the Goethals–Seidel array (4) we obtain more than 3 000 000 skew-Hadamard matrices of order 52. Many of these matrices are expected to be equivalent. In order to check the equivalence of these matrices we used the well known “profile” criterion [3]. We briefly describe how this criterion works.

Suppose H is a Hadamard matrix of order $4n$ with typical entries h_{ij} . We write P_{ijkl} for the absolute value of the generalized inner product of rows i, j, k and ℓ :

$$P_{ijkl} = \left| \sum_{x=1}^{4n} h_{ix} h_{jx} h_{kx} h_{\ell x} \right|.$$

This criterion cannot distinguish all inequivalent Hadamard matrices. For example, it does not work in the case of Hadamard matrices of order $n = 20$ because it gives the same profile for all three equivalent classes of Hadamard matrices of this order. The profile criterion satisfies $P_{ijkl} \equiv 4n \pmod{8}$. We shall write $\pi(m)$ for the number of sets $\{i, j, k, \ell\}$ of four distinct rows such that $P_{ijkl} = m$. From the definition and the above we have that $\pi(m) = 0$ unless $m \geq 0$ and $m \equiv 4n \pmod{8}$. We call $\pi(m)$ the profile (or 4-profile) of H . The (unique) matrices of order 4, 8 and 12 have profiles

$$\begin{aligned} \pi(4) &= 1, \\ \pi(0) &= 56, \quad \pi(8) = 14, \\ \pi(4) &= 495, \quad \pi(12) = 0, \end{aligned}$$

respectively. The five inequivalent classes of order 16 gave four distinct profiles:

$$\begin{aligned} \text{class } H_0: \pi(0) &= 1680, \quad \pi(8) = 0, \quad \pi(16) = 140; \\ \text{class } H_1: \pi(0) &= 1488, \quad \pi(8) = 256, \quad \pi(16) = 76; \\ \text{class } H_2: \pi(0) &= 1392, \quad \pi(8) = 484, \quad \pi(16) = 44; \\ \text{class } H_3: \pi(0) &= 1344, \quad \pi(8) = 448, \quad \pi(16) = 28; \\ \text{class } H_4: \pi(0) &= 1344, \quad \pi(8) = 448, \quad \pi(16) = 28. \end{aligned}$$

The matrices of class H_4 are the transposes of the matrices of class H_3 . The three classes of order 20 all gave the same profile

$$\pi(4) = 4560, \quad \pi(12) = 285, \quad \pi(20) = 0.$$

Even though two inequivalent Hadamard matrices may have the same profile, if two Hadamard matrices have different profile they are surely inequivalent. So, we shall use this criterion to check the equivalence of the derived skew-Hadamard matrices and then we apply the “IsHadamardEquivalent” function of Magma ver.2.10-17 to separate any remaining inequivalent matrices.

Using the above criterion we have found out that there are 552 skew-Hadamard matrices with different profiles. These matrices and their profiles are given in the web page “http://www.math.ntua.gr/people/ckoukouv/en_index.html”. When we apply the Magma software ver.2.10-17 we have found 9 more inequivalent matrices. These matrices are also given in the web page “http://www.math.ntua.gr/people/ckoukouv/en_index.html”. Thus, the lower bound for the number of the inequivalent skew-Hadamard matrices of order 52 is 561.

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References

- [1] A.H. Baartmans, C. Lin, W.D. Wallis, Symmetric and skew equivalence of Hadamard matrices of order 28, *Ars. Combin.* 41 (1995) 25–31.
- [2] D. Blatt, G. Szekeres, A skew Hadamard matrix of order 52, *Canad. J. Math.* 21 (1969) 1319–1322.
- [3] J. Cooper, J. Milas, W.D. Wallis, Hadamard equivalence, *Combinatorial Mathematics, Lecture Notes in Mathematics*, vol. 686, Springer, Berlin, Heidelberg, New York, 1978, pp. 126–135.
- [4] R. Craigen, Hadamard matrices and designs, *The CRC Handbook of Combinatorial Designs*, in: C.J. Colbourn, J.H. Dinitz (Eds.), CRC Press, Boca Raton, FL, (1996), 370–377.
- [5] D.Z. Djokovic, Skew Hadamard matrices of order 4×37 and 4×43 , *J. Combin. Theory Ser. A* 61 (1992) 319–321.
- [6] D.Z. Djokovic, Ten new orders for Hadamard matrices of skew type, *Univ. Beograd. Publ. Electrotehn. Fak. Ser. Math.* 3 (1992) 47–59.
- [7] D.Z. Djokovic, Construction of some new Hadamard matrices, *Bull. Austral. Math. Soc.* 45 (1992) 327–332.
- [8] D.Z. Djokovic, Good matrices of order 33, 35 and 127 exist, *J. Combin. Math. Combin. Comput.* 14 (1993) 145–152.
- [9] D.Z. Djokovic, Five new orders for Hadamard matrices of skew type, *Australasian J. Combin.* 10 (1994) 259–264.
- [10] D.Z. Djokovic, Six new orders for G-matrices and some new orthogonal designs, *J. Combin. Inform. System Sci.* 20 (1995) 1–7.
- [11] D.Z. Djokovic, Skew-Hadamard matrices of order 188 and 388 exist, submitted.
- [12] C. Elster, A. Neumaier, Screening by conference designs, *Biometrika* 82 (1995) 589–602.
- [13] R.J. Fletcher, C. Koukouvinos, J. Seberry, New skew-Hadamard matrices of order $4 \cdot 49$ and new D-optimal designs of order $2 \cdot 59$, *Discrete Math.* 286 (2004) 251–253.
- [14] S. Georgiou, C. Koukouvinos, On circulant G-matrices, *J. Combin. Math. Combin. Comput.* 40 (2002) 205–225.
- [15] S. Georgiou, C. Koukouvinos, Some results on orthogonal designs and Hadamard matrices, *Int. J. Appl. Math.* 17 (2005) 433–443.
- [16] S. Georgiou, C. Koukouvinos, J. Seberry, On circulant best matrices and their applications, *Linear and Multilinear Algebra* 48 (2001) 263–274.
- [17] S. Georgiou, C. Koukouvinos, S. Stylianou, On good matrices, skew Hadamard matrices and optimal designs, *Comput. Statist. Data Anal.* 41 (2002) 171–184.
- [18] S. Georgiou, C. Koukouvinos, S. Stylianou, New skew Hadamard matrices and their application in edge designs, *Utilitas Math.* 66 (2004) 121–136.
- [19] S. Georgiou, C. Koukouvinos, S. Stylianou, Construction of new skew Hadamard matrices and their use in screening experiments, *Comput. Statist. Data Anal.* 45 (2004) 423–429.
- [20] A.V. Geramita, J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York, Basel, 1979.
- [21] J.M. Goethals, J.J. Seidel, A skew Hadamard matrix of order 36, *J. Austral. Math. Soc.* 11 (1970) 343–344.
- [22] H. Kharaghani, B. Tayfeh-Rezaie, A Hadamard matrix of order 428, *J. Combin. Designs* 13 (2005) 435–440.
- [23] C. Koukouvinos, J. Seberry, On G-matrices, *Bull. ICA* 9 (1993) 40–44.
- [24] S. Kounias, T. Chadjipantelis, Some D-optimal weighing designs for $n \equiv 3 \pmod{4}$, *J. Statist. Plann. Inference* 8 (1983) 117–127.
- [25] R.E.A.C. Paley, On orthogonal matrices, *J. Math. Phys.* 12 (1933) 311–320.
- [26] J. Seberry Wallis, (v, k, λ) configurations and Hadamard matrices, *J. Austral. Math. Soc.* 11 (1970) 297–309.
- [27] J. Seberry Wallis, A skew-Hadamard matrix of order 92, *Bull. Austral. Math. Soc.* 5 (1971) 203–204.
- [28] J. Seberry Wallis, Some $(1, -1)$ matrices, *J. Combin. Theory Ser. B* 10 (1971) 1–11.
- [29] J. Seberry Wallis, On supplementary difference sets, *Aequationes Math.* 8 (1972) 242–257.
- [30] J. Seberry Wallis, On Hadamard matrices, *J. Combin. Theory Ser. A* 18 (1975) 149–164.
- [31] J. Seberry Wallis, Construction of Williamson type matrices, *Linear and Multilinear Algebra* 3 (1975) 197–207.
- [32] J. Seberry Wallis, On skew Hadamard matrices, *Ars Combin.* 6 (1978) 255–275.
- [33] J. Seberry Wallis, A.L. Whiteman, Some classes of Hadamard matrices with constant diagonal, *Bull. Austral. Math. Soc.* 7 (1972) 233–249.
- [34] J. Seberry, M. Yamada, Hadamard matrices, sequences and block designs, in: J.H. Dinitz, D.R. Stinson (Eds.), *Contemporary Design Theory—A Collection of Surveys*, Wiley, New York, 1992, pp. 431–560.
- [35] E. Spence, Skew-Hadamard matrices of order $2(q+1)$, *Discrete Math.* 18 (1977) 79–85.
- [36] G. Szekeres, Cyclotomy and complementary difference sets, *Acta Arith.* 18 (1971) 349–353.
- [37] G. Szekeres, A note on skew type orthogonal ± 1 matrices, in: A. Hajnal, L. Lovász, V.T. Sós (Eds.), *Combinatorics, Colloquia Mathematica Societatis, Janos Bolyai*, vol. 52, North-Holland, Amsterdam, 1988, pp. 489–498.
- [38] W.D. Wallis, A.P. Street, J. Seberry Wallis, *Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices*, *Lecture Notes in Mathematics*, vol. 292, Springer, Berlin, Heidelberg, New York, 1972.
- [39] A.L. Whiteman, An infinite family of skew-Hadamard matrices, *Pacific J. Math.* 38 (1971) 817–822.
- [40] A.L. Whiteman, Skew Hadamard matrices of Goethals–Seidel type, *Discrete Math.* 2 (1972) 397–405.
- [41] X.-M. Zhang, G-matrices of order 19, *Bull. ICA* 4 (1992) 95–98.